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INVENTORY CONTROL WITH TWO SWITCH-OVER LEVELS FOR A CLASS OF M/G/1 QUEUEING SYSTEMS WITH VARIABLE ARRIVAL AND SERVICE RATE

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This paper deals with inventory control in a class of M/G/1 queueing systems. At each point of time the system can be switched from one of two possible stages to another. The rate of arrival process and the service rate depend on the stage of the system. The cost structure imposed on the model includes both fixed switch-over costs and a holding cost at a general rate depending on the stage of the system. The rule for controlling the inventory is specified by two switch-over levels.

Using an embedding approach, we will derive a formula for the long-run average expected costs per unit time of this policy. By an appropriate choice of the cost parameters, we may obtain various operating characteristics for the system amongst which the stationary distribution of the inventory and the average number of switch-overs per unit time.

M/G/1 queue	variable arrival and service rate
inventory control	two switch-over levels
general cost structure	average costs
stationary distribution	

1. Introduction

1.1. Model

This paper deals with inventory control in a class of M/G/1 queueing systems which are considered as inventory systems with the inventory at time t being the virtual waiting time. The system is supposed to have a finite capacity K. It is assumed that at each point of time the system is in one of two possible stages 1 and 2 where at any moment the system can be switched from one stage to another without loss of time. If the system is in stage i, the epochs at which customers arrive are generated by a Poisson process with rate λ_i, $i = 1, 2$. Let Y_i be a positive random variable having probability distribution function $F_i(x) = \mathrm{P}\{Y_i \leq x\}$, $i = 1, 2$. Any customer, arriving while the system is in stage i and the inventory of the system is x, enlarges the inventory with an amount which is distributed as $\min[K-x, Y_i]$ and causes an overflow which is distributed as $\max[0, Y_i - K + x]$, $i = 1,2$. If the system is in stage i and the inventory is positive, then between arrival epochs the inventory decreases linearly at rate $\sigma_i > 0$, $i = 1,2$.

The following cost structure is imposed on the model. There are holding (and service) costs at rate $h_i(x)$ when inventory is x and the system is in stage i where the functions $h_1(x)$ and $h_2(x)$ are assumed to be bounded functions having only a finite number of discontinuities in $0 \leq x \leq K$. An overflow cost of $p_i(y)$ is incurred when an overflow of an amount y is caused by a customer arriving while the system is in stage i where $p_i(y)$ is a nondecreasing function of $y \geq 0$ with $\int_0^\infty p_i(y)\mathrm{d}F_i(y) < \infty$ for $i = 1,2$. Finally, a fixed cost of γ is incurred when the system is switched from stage 2 to stage 1.

1.2. The control rule

The rule for controlling the inventory is specified by two switch-over levels y_1 and y_2 with $0 \leq y_2 \leq y_1 < K$. This (y_1, y_2) policy prescribes to switch the system from stage 1 to stage 2 only when the inventory exceeds the value y_1 and prescribes to switch the system from stage 2 to stage 1 only when the inventory has been decreased to the value y_2.

Using a powerful and simple approach involving embedded processes, we shall derive a formula for the long-run average expected costs per unit time of this policy. By an appropriate choice of the cost functions, we may obtain from this formula various operating characteristics for the system amongst which the stationary distribution of the inventory and the average number of switch-overs and overflows per unit time.

The above control problem includes as special cases a variety of problems previously studied in the literature and provides thus a unifying treatment of these problems. As examples we give the following two cases.

Case (i) $\lambda_1 = \lambda_2$, $F_1(.) = F_2(.)$. In this case the control of the inventory is achieved by controlling the service rate. This problem was studied amongst others in [3], [10] and [11]. In [3] the stationary distribution of the inventory was derived for the infinite capacity model with a single switch-over level $y_1 = y_2$ (see also [10]) and in [11] the average cost of the (y_1, y_2) policy was obtained for the infinite capacity model for the case of linear holding costs.

Case (ii) $\lambda_1 > 0$, $\lambda_2 = 0$, $F_1(.) = F_2(.)$, $\sigma_1 = \sigma_2$. In this case we have in fact a queueing system with restricted accessibility and the control of the inventory is achieved by controlling the arrival process. For this model with an infinite capacity and a single switch-over level $y_1 = y_2$ the stationary distribution of the inventory was derived in [1], see also [4] for this specific model. Finally, we observe that by letting $y_1 = y_2$ and $y_1 \to K$ the model of case (ii) reduces to the well-known finite dam model.

2. Preliminaries and the embedding approach

In this section we first introduce some notation that will be needed hereafter. Next we give the generally applicable embedding approach which underlies the derivation of the formula for the long-run average costs.

For $i = 1,2$, let $H_i(x) = 0$ for $x < 0$ and let

$$H_i(x) = \frac{\lambda_i}{\sigma_i} \int_0^x \{1 - F_i(y)\} \mathrm{d}y \quad \text{for } x \geq 0.$$

Next we define δ_i as the unique number satisfying

$$\int_0^\infty \mathrm{e}^{-\delta_i y} \, \mathrm{d}H_i(y) = 1, \quad i = 1,2.$$

For $i = 1,2$, define the probability distribution function G_i by $G_i(x) = 0$ for $x < 0$ and

$$G_i(x) = \int_0^x \mathrm{e}^{-\delta_i y} \, \mathrm{d}H_i(y) \quad \text{for } x \geq 0.$$

Finally define for $i = 1,2$ the renewal function M_i by $M_i(x) = 0$ for $x < 0$ and

$$M_i(x) = \sum_{n=1}^{\infty} G_i^n(x) \quad \text{for } x \geq 0,$$

where G_i^n is the n-fold convolution of G_i with itself. Fix $1 \leq i \leq 2$ and $w > 0$. Consider now the following "renewal-type" equation

$$u(x) = a(x) + \int_0^{w-x} u(x+y) \mathrm{d}H_i(y), \quad 0 \leq x \leq w, \tag{1}$$

where $a(x)$ is a given bounded Baire function in $0 \leq x \leq w$. This equation can be equivalently written as (cf. p. 77 in [2] and p. 362 in [5]),

$$\mathrm{e}^{\delta_i x} u(x) = \mathrm{e}^{\delta_i x} a(x) + \int_0^{w-x} \mathrm{e}^{\delta_i (x+y)} u(x+y) \mathrm{d}G_i(y), \quad 0 \leq x \leq w.$$

From the solution of this renewal equation for $u^*(x) = \mathrm{e}^{\delta_i x} u(x)$ we find that (1) has the unique bounded solution (cf. [2] and [5])

$$u(x) = a(x) + \int_0^{w-x} \mathrm{e}^{\delta_i y} a(x+y) \mathrm{d}M_i(y), \quad 0 \leq x \leq w. \tag{2}$$

We now give the following relation, cf. p. 77 in [2]. Fix $1 \leq i \leq 2$ and $w > 0$, and let $a(\,.\,)$ be any continuous function. Using the fact that $F_i(0) = 0$, we have for each point x such that F_i is continuous at $w - x$,

$$\frac{\partial}{\partial x} \int_0^{w-x} a(x+y)\{1 - F_i(y)\} \mathrm{d}y = -a(x) + \int_0^{w-x} a(x+y) \mathrm{d}F_i(y). \tag{3}$$

After these preliminaries, let us now consider the queueing problem and let us define the state of the system as $x(x')$ when the inventory level is x and the system is in stage 1(2). Denote by $X(t)$ and $S(t)$ the inventory level and the state of the system at time t respectively where we take the processes $\{X(t), t \geq 0\}$ and $\{S(t), t \geq 0\}$ continuous from the right. To derive the formula for the average cost, we will study a properly chosen embedded Markov chain of the process $\{S(t)\}$. Consider now the inventory system controlled by a fixed (y_1, y_2) policy where for notational convenience we take $y_2 > 0$. Unless stated otherwise, we also assume for ease that the system is empty at epoch 0. Now, let $T_0 = 0$ and, for $n \geq 1$, let T_n be the nth epoch at which either the inventory level exceeds y_1 while the system is in stage 1 or the inventory level decreases to y_2 while the system is in stage 2 or the inventory becomes zero. For any $n \geq 0$, define S_n as the state of the system at epoch T_n. The embedded discrete-time process $\{S_n, n = 0, 1, \ldots\}$ is a Markov chain with state space

$$S = \{0\} \cup \{y_2\} \cup \{x' \mid y_1 < x \leq K\}.$$

Taking for ϕ any finite measure on the Borel sets of S such that $\phi(A) > 0$ if and only if $0 \in A$, it is easily verified that for any A with $\phi(A) > 0$ we have $\mathbf{P}\{S_m \in A$ for some $1 \leq m \leq n \mid S_0 = s\} \to 1$ uniformly in $s \in S$ as $n \to \infty$, which says that the Markov chain $\{S_n\}$ is uniformly ϕ-recurrent, see p. 25 in [7]. Now, by Theorem 7.1 in [7], the Markov chain $\{S_n\}$ has a unique invariant probability measure π such that, for any Borel subset A of S,

$$\pi(A) = \int_S P(s, A)\pi(\mathrm{d}s) \tag{4}$$

where $P(.,.)$ denotes the one-step transition probability distribution function of $\{S_n\}$. Alternatively, the result (4) may also be obtained from the fact that for the Markov chain $\{S_n\}$ the mean recurrence time from state s to state 0 is finite for all $s \in S$ (e.g. Theorem 1 in [11]). Moreover, using the fact that $\{S_n\}$ is uniformly ϕ-recurrent, we have by Theorem 3.3 in [6]

$$\lim_{n\to\infty} \frac{1}{n} \sum_{k=0}^{n-1} \mathbf{E} f(S_k) = \int_S f(s)\pi(\mathrm{d}s) \tag{5}$$

for any Baire function f such that $\int |f(s)| \pi(\mathrm{d}s)$ is finite.

Define $Z(t)$ as the total costs incurred in $(0, t]$. For any $n \geq 0$, let Z_n be the total costs incurred in $(T_n, T_{n+1}]$. Also let

$$c(s) = \mathbf{E}(Z_n \mid S_n = s) \quad \text{and} \quad \tau(s) = \mathbf{E}(T_{n+1} - T_n \mid S_n = s) \quad \text{for } s \in S.$$

Since the process $\{S(t)\}$ is regenerative with the epochs at which the system becomes empty as regeneration epochs, it follows from the proof of Theorem 7.5 in [8] that

$$\lim_{t\to\infty} \frac{1}{t} \mathbf{E}\, Z(t) = \lim_{n\to\infty} \sum_{k=0}^{n-1} \mathbf{E}\, Z_k \Big/ \sum_{k=0}^{n-1} \mathbf{E}(T_{k+1} - T_k).$$

Hence, by (5), the average cost of the (y_1, y_2) policy is given by

$$\lim_{t\to\infty} \frac{1}{t} \mathbf{E}\, Z(t) = \int_S c(s)\pi(\mathrm{d}s) \Big/ \int_S \tau(s)\pi(\mathrm{d}s). \tag{6}$$

3. Average costs and various operating characteristics

In this section we shall first determine the stationary distribution π and the functions c and τ. These quantities have been defined for the embedded Markov chain $\{S_n\}$ and, by formula (6), they give the long-run average expected costs per unit time for the continuous-time queueing problem. By an appropriate choice of the cost parameters, we next find various operating characteristics for the system such as the stationary distribution of the inventory, the average number of overflows per unit time and the average number of switch-overs per unit time. The results we will obtain involve the renewal functions M_1 and M_2. For the special case in which the service time distributions are exponential these renewal functions can be explicitly given.

To determine the stationary distribution π, define for all $0 \leqslant u \leqslant y_1$ and $y_1 \leqslant v \leqslant K$,

$p(u, v)$ = probability that the first value of the process $\{X(t), t \geqslant 0\}$ taken on in the set $\{0\} \cup \{x \mid y_1 < x \leqslant K\}$ belongs to the set $\{x \mid v \leqslant x \leqslant K\}$ given that $X(0) = u$,

and let $p_0(u) = 1 - p(u, y_1)$ for $0 \leqslant u \leqslant y_1$. For ease of notation, write

$$\pi_0 = \pi(\{0\}), \qquad \pi_2 = \pi(\{y_2\}),$$

$$\pi(v) = \pi(\{x' \mid v \leqslant x \leqslant K\}, \quad y_1 \leqslant v \leqslant K.$$

Also, let $\bar{F}_1(y) = 1 - \lim_{x \uparrow y} F_1(x) = \mathbf{P}\{Y_1 \geqslant y\}$. Then, by (4),

$$\pi(v) = \pi_0 \left\{ \bar{F}_1(v) + \int_0^{y_1} p(y, v) \mathrm{d}F_1(y) \right\} + \pi_2 p(y_2, v), \qquad y_1 \leqslant v \leqslant K, \tag{7}$$

$$\pi_0 = \pi_0 \int_0^{y_1} p_0(y) \mathrm{d}F_1(y) + \pi_2 p_0(y_2) \quad \text{and} \quad \pi_2 = \pi(y_1). \tag{8}$$

We note that any interval of integration is closed, unless stated otherwise. Together (7), (8) and the relation $\pi_0 + \pi_2 + \pi(y_1) = 1$ determine the stationary distribution π once we have calculated the probabilities $p(u, v)$.

Now fix v with $y_1 \leqslant v \leqslant K$. We first observe that $p(u, v)$ is continuous in $0 \leqslant u \leqslant y_1$. Next, by using standard arguments, we have for all $0 < u < y_1$,

$$p(u + \Delta u, v) = \lambda_1 \frac{\Delta u}{\sigma_1} \left\{ \bar{F}_1(v - u) + \int_0^{y_1 - u} p(u + y, v) \mathrm{d}F_1(y) \right\} + \left(1 - \lambda_1 \frac{\Delta u}{\sigma_1}\right) p(u, v) + \mathrm{o}(\Delta u),$$

from which we get for all $0 < u < y_1$,

$$\frac{\partial p(u, v)}{\partial u} = \frac{\lambda_1}{\sigma_1} \left\{ \bar{F}_1(v - u) - p(u, v) + \int_0^{y_1 - u} p(u + y, v) \mathrm{d}F_1(y) \right\}.$$

Using (3) and the continuity of $p(\,.\,,v)$. we easily obtain from this differential equation

$$p(u,v)=\phi(u,v)+\int_0^{y_1-u} p(v+y,v)\,\mathrm{d}H_1(y) \quad \text{for } 0\leqslant u\leqslant y_1$$

where, for some constant c_v and for all $0\leqslant u\leqslant y_1$,

$$\phi(u,v)=c_v+\frac{\lambda_1}{\sigma_1}\int_0^u \bar{F}_1(v-y)\,\mathrm{d}y=c_v+H_1(v)-H_1(v-u).$$

Next, by (1) and (2), we get

$$p(u,v)=\phi(u,v)+\int_0^{y_1-u} \mathrm{e}^{\delta_1 y}\phi(u+y,v)\,\mathrm{d}M_1(y) \quad \text{for } 0\leqslant u\leqslant y_1.$$

For any v with $y_1\leqslant v\leqslant K$ the constant c_v follows from the boundary condition

$$p(0,v)=0.$$

This completes the determination of the stationary distribution π. To determine the functions $c(\,.\,)$ and $\tau(\,.\,)$, we define for all $0\leqslant x\leqslant y_1$,

$k_1(x)=$ the expected holding and overflow costs incurred up to the first epoch at which the process $\{S(t), t\geqslant 0\}$ takes on a state in the set $\{0\}\cup\{y'\,|\,y_1<y\leqslant K\}$ given that $S(0)=x$,

and, for all $y_2\leqslant x\leqslant K$,

$k_2(x)=$ the expected holding and overflow costs incurred up to the first epoch at which the process $\{S(t), t\geqslant 0\}$ takes on the state y_2 given that $S(0)=x'$.

It is now easily seen that the function $c(s)$, $s\in S$ is given by

$$c(0)=\frac{h_1(0)}{\lambda_1}+\int_0^{y_1} k_1(y)\,\mathrm{d}F_1(y)+\int_K^{\infty} p_1(y-K)\,\mathrm{d}F_1(y),$$

$$c(y_2)=k_1(y_2) \quad \text{and} \quad c(x')=k_2(x)+\gamma \quad \text{for } y_1<x\leqslant K.$$

Clearly, for any $s\in S$ the formula for $\tau(s)$ follows from the corresponding one for $c(s)$ by putting $h_i(x)=1$ for $x\geqslant 0$, $p_i(y)=0$ for $y\geqslant 0$ $(i=1,2)$ and $\gamma=0$. The functions $k_1(x)$ and $k_2(x)$ will be determined in a very similar way as $p(u,v)$. First observe that these functions are continuous in $0\leqslant x\leqslant K$. Then, for any $0<x<y_1$ such that x is a continuity point of h_1

$$\begin{aligned}k_1(x+\Delta x)=h_1(x)\frac{\Delta x}{\sigma_1}&+\lambda_1\frac{\Delta x}{\sigma_1}\left\{\int_0^{y_1-x} k_1(x+y)\,\mathrm{d}F_1(y)\right.\\&\left.+\int_{K-x}^{\infty} p_1(x+y-K)\,\mathrm{d}F_1(y)\right\}\\&+\left(1-\lambda_1\frac{\Delta x}{\sigma_1}\right)k_1(x)+\mathrm{o}(\Delta x),\end{aligned}$$

from which we get for any $0 < x < y_1$ such that x is a continuity point of h_1,

$$k_1'(x) = \frac{h_1(x)}{\sigma_1} + \frac{\lambda_1}{\sigma_1} \int_{K-x}^{\infty} p_1(x+y-K)\mathrm{d}F_1(y) - \frac{\lambda_1}{\sigma_1} k_1(x) + \frac{\lambda_1}{\sigma_1} \int_0^{y_1-x} k_1(x+y)\mathrm{d}F_1(y).$$

In the same way as above we get from this differential equation that, for some constant b_1,

$$k_1(x) = d_1(x) + b_1 + \int_0^{y_1-x} \mathrm{e}^{\delta_1 y}\{d_1(x+y)+b_1\}\mathrm{d}M_1(y) \quad \text{for } 0 \leq x \leq y_1,$$

where

$$d_1(x) = \int_0^x \left[\frac{h_1(u)}{\sigma_1} + \frac{\lambda_1}{\sigma_1} \int_{K-u}^{\infty} p_1(u+y-K)\mathrm{d}F_1(y)\right]\mathrm{d}u \quad \text{for } 0 \leq x \leq y_1.$$

The constant b_1 is determined by the boundary condition $k_1(0) = 0$. Similarly, we find from the corresponding differential equation for $k_2(x)$ that, for some constant b_2,

$$k_2(x) = d_2(x) + b_2 + \int_0^{K-x} \mathrm{e}^{\delta_2 y}\{d_2(x+y)+b_2\}\mathrm{d}M_2(y) \quad \text{for } y_2 \leq x \leq K.$$

where, for $y_2 \leq x \leq K$,

$$d_2(x) = \int_0^x \left[\frac{h_2(u)}{\sigma_2} + \frac{\lambda_2}{\sigma_2} \int_{K-u}^{\infty} p_2(u+y-K)\mathrm{d}F_2(y) + \right. \left. + \frac{\lambda_2}{\sigma_2}\{1-F_2(K-u)\}k_2(K)\right]\mathrm{d}u.$$

The constant b_2 and the value $k_2(K)$ follow by putting $x = K$ in the above formula for $k_2(x)$ and using the boundary condition $k_2(y_2) = 0$.

We now have completed the determination of π, $c(\,.\,)$ and $\tau(\,.\,)$ and so, by (6), we have determined a formula for the average cost of the (y_1, y_2) policy. From this formula we may obtain various operating characteristics for the system. To obtain the stationary distribution of the inventory, define for any $t \geq 0$ the random variable $A(t) = i$ when the system is in stage i at time t, $i = 1, 2$, where we take the process $\{A(t)\}$ continuous from the right. Fix now k and z with $k = 1, 2$ and $0 \leq z \leq K$, take $h_k(x) = 1$ for $0 \leq x \leq z$, $h_k(x) = 0$ for $x > z$ and take the other holding cost function, the overflow cost functions and the fixed switch-over cost identical to zero. Then, using standard results from the theory of regenerative processes (e.g. [9]), we have

$$\lim_{t\to\infty} \mathbf{P}\{A(t) = k,\, X(t) \leq z\} = \lim_{t\to\infty} \frac{\mathbf{E}\,Z(t)}{t},$$

so the stationary distribution of the inventory is determined by the right-hand side of (6).

Clearly, the average number of switch-overs per unit time is equal to the coefficient of γ in the formula for the average cost and is given by

$$(1-\pi_0-\pi_2)\Big/\int_S \tau(s)\pi(\mathrm{d}s).$$

Finally, letting $p_i(y)=p_i$ for $y>0$ and $p_i(0)=0$, we have that the coefficient of p_i in the formula for the average cost gives the average number of overflows in stage i per unit time, $i=1,2$.

To end, we consider the special case where

$$F_i(x)=1-\mathrm{e}^{-\eta_i x} \quad \text{for } x>0 \text{ and } i=1,2.$$

We then find $\delta_i=(\lambda_i/\sigma_i)-\eta_i$ and

$$\mathrm{e}^{\delta_i y}M_i'(y)=(\lambda_i/\sigma_i)\mathrm{e}^{-(\eta_i-\lambda_i/\sigma_i)y} \quad \text{for } y\geqslant 0 \text{ and } i=1,2.$$

In the remainder it is supposed that $\lambda_i/\sigma_i\neq\eta_i$ for $i=1,2$. Put for abbreviation

$$\alpha_i=\frac{\lambda_i}{\sigma_i}, \qquad \beta_i=\eta_i-\alpha_i \qquad \text{for } i=1,2,$$

$$R(y_1,y_2)=\beta_1^{-1}\{\eta_1\mathrm{e}^{\beta_1 y_1}-\alpha_1\mathrm{e}^{\beta_1 y_2}\}, \qquad S(y_1)=\mathrm{e}^{-\eta_1(K-y_1)}.$$

We find after elementary but lengthy calculations

$$\pi_0=c\,.\,R(y_1,y_2), \qquad \pi_2=c, \qquad -\frac{\mathrm{d}\pi(v)}{\mathrm{d}v}=c\eta_1\mathrm{e}^{-\eta_1(v-y_1)} \quad \text{for } y_1<v<K,$$

$$\pi(K)=c\,.\,S(y_1),$$

where the normalizing constant c equals $1/\{R(y_1,y_2)+2\}$. We next find

$$\begin{aligned}\frac{1}{c}\int_S \tau(s)\pi(\mathrm{d}s)=&\left(\frac{\eta_2}{\sigma_2\beta_2}-\frac{\eta_1}{\sigma_1\beta_1}\right)\left(y_1-y_2+\frac{1}{\eta_1}\right)+\frac{\eta_1}{\lambda_1\beta_1}R(y_1,y_2)+\\ &-\frac{\eta_2}{\sigma_2\eta_1\beta_2}S(y_1)+\frac{\alpha_2\eta_1}{\sigma_2\beta_2^2(\beta_2-\eta_1)}\{\mathrm{e}^{-\beta_2(K-y_1)}-S(y_1)\}+\\ &+\frac{\alpha_2}{\sigma_2\beta_2^2}\{\mathrm{e}^{-\beta_2(K-y_2)}-S(y_1)\}.\end{aligned}$$

Denote by $D(y_1,y_2)$ the right-hand side of this equation. We then obtain

$$\lim_{t\to\infty}\mathbf{P}\{A(t)=1,\,X(t)\leqslant z\}=\frac{1}{D(y_1,y_2)}\frac{R(y_1,y_2)}{\lambda_1\beta_1}\{\eta_1-\alpha_1\mathrm{e}^{-\beta_1 z}\},\quad 0\leqslant z\leqslant y_2,$$

$$\lim_{t\to\infty}\mathbf{P}\{A(t)=1,\,X(t)\leqslant z\}=$$

$$= \frac{1}{D(y_1, y_2)} \left[\frac{\eta_1}{\sigma_1 \beta_1} (y_2 - z) + \frac{\eta_1}{\lambda_1 \beta_1} R(y_1, y_2) + \frac{1}{\sigma_1 \beta_1 2} (\alpha_1 - \eta_1 e^{-\beta_1 (z - y_1)}) \right], \qquad y_2 \leqslant z \leqslant y_1,$$

$$\lim_{t \to \infty} \mathbf{P}\{A(t) = 2, X(t) \leqslant z\}$$
$$= \frac{1}{D(y_1, y_2)} \left[\frac{\eta_2}{\sigma_2 \beta_2} (z - y_2) + \frac{\alpha_2}{\sigma_2 \beta_2^2} (e^{-\beta_2 (z - y_2)} - 1) \right], \qquad y_2 \leqslant z \leqslant y_1,$$

$$\lim_{t \to \infty} \mathbf{P}\{A(t) = 2, X(t) \leqslant z\} =$$
$$= \frac{1}{D(y_1, y_2)} \left[\frac{\eta_2}{\sigma_2 \beta_2} \left(y_1 - y_2 + \frac{1}{\eta_1} \right) + \frac{\alpha_2}{\sigma_2 \beta_2^2} e^{-\beta_2 (z - y_2)} + \right.$$
$$+ \frac{\eta_1 \alpha_2}{\sigma_2 \beta_2^2 (\beta_2 - \eta_1)} \{ e^{-\beta_2 (z - y_1)} - e^{-\eta_1 (z - y_1)} \} +$$
$$\left. - \frac{1}{\sigma_2 \beta_2} \left(\frac{\eta_2}{\eta_1} + \frac{\alpha_2}{\beta_2} \right) e^{-\eta_1 (z - y_1)} \right], \qquad y_1 \leqslant z \leqslant K.$$

Further, the average number of overflows in stage 1 per unit time equals

$$S(y_1)/D(y_1, y_2)$$

and the average number of overflows in stage 2 per unit time is equal to

$$\frac{1}{D(y_1, y_2)} \left[\frac{\alpha_2}{\beta_2} \{ S(y_1) - e^{-\beta_2 (K - y_2)} \} + \frac{\alpha_2 \eta_1}{\beta_2 (\beta_2 - \eta_1)} \{ S(y_1) - e^{-\beta_2 (K - y_1)} \} \right].$$

Finally, letting $\lambda_2 = 0$, $\sigma_1 = \sigma_2 = 1$, $y_1 = y_2 = N$ and $K \to \infty$, we find

$$\lim_{t \to \infty} \mathbf{P}\{X(t) \leqslant z\} = \begin{cases} c_1 [1 - (\lambda_1/\eta_1) e^{-\beta_1 z}], & 0 \leqslant z \leqslant N, \\ c_1 [1 - (\lambda_1/\eta_1)^2 e^{-\beta_1 N} - (\lambda_1/\eta_1)(1 - \lambda_1/\eta_1) e^{-\eta_1 z + \lambda_1 N}], & z \geqslant N \end{cases}$$

where $c_1 = 1/[1 - (\lambda_1/\eta_1)^2 e^{\beta_1 N}]$. This formula corrects a slight error in a corresponding formula in [1].

References

[1] J.W. Cohen, Single server queues with restricted accessibility, Journal of Engineering Mathematics 3 (1969) 265–284.

[2] J.W. Cohen, On Regenerative Processes in Queueing Theory, Lecture Notes in Economics and Mathematical Systems 121 (Springer, Berlin, 1976).

[3] J.W. Cohen, On the optimal switching level for an M/G/1 queueing system, Stochastic Processes and their Applications 4 (1976) 297–316.

[4] B.T. Doshi, Continuous time control of Markov processes on an arbitrary state space, Technical Summary Report no. 1468, University of Wisconsin, Wisconsin (1974).

[5] W. Feller, An Introduction to Probability Theory and its Applications, Vol. II (Wiley, New York, 1966).
[6] N.C. Jain, Some limit theorems for general Markov processes, Z. Wahrscheinlichkeitstheorie verw. Geb. 6 (1966) 206–223.
[7] S. Orey, Limit Theorems for Markov Chain Transition Probabilities (Van Nostrand Reinhold Company, London, 1971).
[8] S.M. Ross, Applied Probability Models with Optimization Applications (Holden-Day, Inc., San Francisco, 1970).
[9] S. Stidham, Jr., Regenerative processes in the theory of queues, with applications to the alternating-priority queue, Advances in Applied Probability 4 (1972) 542–557.
[10] R.M. Thatcher, Optimal single-channel service policies for stochastic arrivals, Report ORC 68-16, Operations Research Center, University of California, Berkeley 1968.
[11] H.C. Tijms, On a switch-over policy for controlling the workload in a queueing system with two constant service rates and fixed switch-over costs, Zeitschrift für Operations Research 21 (1977) 19–32.